A STATISTICAL PHYSICS

The purpose of this script is an introduction into the fundamental concepts of thermodynamics and how they are determined from a microscopic model of matter. This includes especially the concept of the state variables temperature, pressure and chemical potential, and how changes of entropy, volume and particle number introduce changes in the internal energy; as well as how well thermal energy can be used in mechanical devices. Restricting ourselves to thermal equilibria where a notion of temperature exists, it is possible to set up partition sums over the microscopic states that a system might be in and to link these partition sums to thermodynamic potentials: Ideally, this bridges the gap between thermodynamics as a continuum theory with no particular assumption about the microscopic properties of a system, and the partition sum as a reflection of exactly these microscopic properties. Thermodynamics and statistical physics as the fundamental theory behind it are incredibly general and require temperature as a concept, joined with occupation statistics of the respective states of a system and a counting scheme for these states, whether they form a continuum or are discrete. We will end this lecture by looking at the phenomena of statistical systems like magnetism or Bose-Einstein-condensation and the dynamics of statistical systems with the Langevin-equation and the Fokker-Planck-equation. Perhaps the best way to view the relation between thermodynamics and statistical physics is that of an effective field theory: Thermodynamics describes systems without any recourse to their actual microscopic structure, which makes concepts like entropy so difficult to understand.

Statistical physics is a branch of physics that uses methods of probability theory and statistics, and particularly the mathematical tools for dealing with large populations and approximations, in solving physical problems. It can describe a wide variety of physical systems with an inherently stochastic nature on the microscopic level. Its applications include many problems in the fields of physics, biology, chemistry, neuroscience, and even some social sciences, such as sociology and linguistics. Its main purpose is to clarify the properties of matter in aggregate, in terms of physical laws governing atomic motion.

⚠ *statistical physics is so incredibly general that you find applications in all branches of physics including gravity*

Without any idea about atoms and molecules in the time before Ludwig Boltzmann one would have imagined the atmosphere to be a continuum described by field quantities like density ρ, pressure p and velocity $\boldsymbol{\upsilon}$, obeying the Euler-equation

$$\partial_t \boldsymbol{\upsilon} + (\boldsymbol{\upsilon} \cdot \nabla)\boldsymbol{\upsilon} = -\frac{\nabla p}{\rho} - \nabla\Phi \tag{A.1}$$

relating the accelerations of the fluid elements to gradients in pressure p and gravitational potential Φ, for an ideal fluid. Assuming that the atmosphere is static, $\boldsymbol{\upsilon} = 0$, and stationary, $\partial_t \boldsymbol{\upsilon} = 0$, results in the hydrostatic equation

$$\frac{\nabla p}{\rho} = -\nabla\Phi \tag{A.2}$$

For continuing, one needs a (possibly phenomenological) relation between pressure and density, i.e. an equation of state. If there is a proportionality $p \sim \rho$ one gets

$$\frac{\nabla p}{\rho} \sim \frac{\nabla \rho}{\rho} = \nabla \ln \rho = -\nabla\Phi \quad \rightarrow \quad \rho \sim \exp(-\Phi). \tag{A.3}$$

Assuming that the gravitational potential is homogeneous,

$$\boldsymbol{g} = -\nabla\Phi = \mathrm{const} \quad \rightarrow \quad \Phi = gh \quad \rightarrow \quad \rho \sim \exp(-h) \tag{A.4}$$

i.e. the barometric formula with an exponential decrease of density with height h.

The reasoning in statistical physics is very different: Matter is not a continuum but made of discrete particles, which experience thermal fluctuations in their energy, due to a continuous reshuffling of energy between all mutually interacting degrees of freedom. In thermal equilibrium, however, it is possible to write down the probability of a thermal fluctuation of a certain size, i.e. the probability that the energy ϵ is borrowed from the system by a single particle or degree of freedom: According to Ludwig Boltzmann, this probability is given by

$$p(\epsilon) \sim \exp\left(-\frac{\epsilon}{k_\mathrm{B}\mathrm{T}}\right) \tag{A.5}$$

such that large fluctuations in energy are rare, but become less rare when the temperature T is increased. After a particle has borrowed the energy ϵ, it can climb in the gravitational field of the Earth to a height $\epsilon \propto gh$, and the fraction $p(\epsilon)$ of all particles must be the density at height h

$$\rho \sim \exp(-h) \tag{A.6}$$

While the result is certainly consistent with the one from continuum mechanics, it seems to involve a lot of intuition. In particular, eqn. (A.5) makes a deep statement about the probability of a thermal fluctuation to occur: Clearly only applicable in thermal equilibrium and for systems with a defined temperature but without any specification of the internal structure of the system there is a universal probability distribution of a rather simple shape. The only parameter in eqn. (A.5) is the Boltzmann-constant k_B with the numerical value

$$k_\mathrm{B} \simeq 1.3806503 \times 10^{-23}\,\frac{\mathrm{J}}{\mathrm{K}} \tag{A.7}$$

which is an incredibly tiny number: Macroscopic objects do not move spontaneously, in contract to microscopic objects such as atoms or molecules, which are in a state of constant motion. At room temperature $\mathrm{T} \simeq 300\mathrm{K}$ there is typically a fluctuating thermal energy of $\epsilon = k_\mathrm{B}\mathrm{T} = 4.2 \times 10^{-21}\mathrm{J}$, which is irrelevant compared to e.g. typical kinetic or potential energies of a macroscopic object. Fig. 1 and 2 illustrate the Boltzmann-probability as a function of energy and temperature (in a choice of units where $k_\mathrm{B} = 1$.

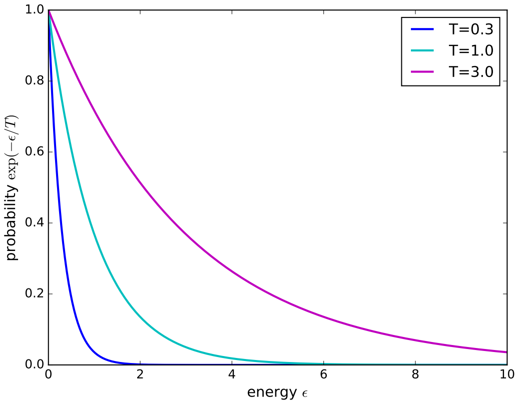

Figure 1: Boltzmann-probability exp(−ϵ/T) as a function of energy and for three different temperatures.

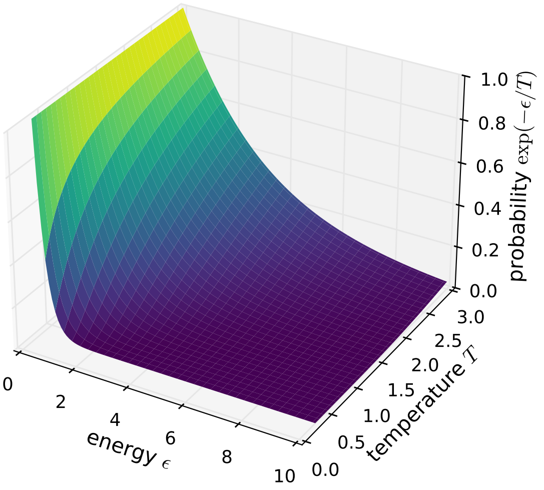

Figure 2: Boltzmann-probability exp(−ϵ/T) as a function of energy and temperature.